Riemann-Stieltjes Integration

The classical *Riemann integral* (Bernhard Riemann 1826-1866) of ordinary calculus has many applications including the computation of areas between two curves, volumes of solids of revolution, arc length, work performed by a variable force, hydrostatic pressures, moments and centers of mass, consumers' surplus, present value of future income, expected values, and variance. In this unit we investigate a generalization of the *Riemann integral* called the *Riemann-Stieltjes integral* (Thomas Joannes Stieltjes 1856-1894) . That is, a special case of the *Riemann-Stieltjes integral* is the *Riemann integral* of ordinary calculus.

Definition 75

Let f be a bounded function on $[a, b]$, α a nondecreasing function on $[a, b]$, and $\mathcal{P}$ a partition of $[a, b]$. If

$$M_k = \underset{x \in [x_{k-1}, x_k]}{lub} f(x) \quad (k = 1, 2, 3, \ldots, n)$$

and

$$m_k = \underset{x \in [x_{k-1}, x_k]}{glb} f(x) \quad (k = 1, 2, 3, \ldots, n),$$

then the ***upper Riemann-Stieltjes sum of f with respect to α on*** $[a, b]$ is defined by

$$RS^+(f, \alpha, \mathcal{P}) = \sum_{k=1}^{n} M_k \left(\alpha(x_k) - \alpha(x_{k-1}) \right)$$

and ***lower Riemann-Stieltjes sum of f with respect to α on*** $[a, b]$ is defined by

$$RS^-(f, \alpha, \mathcal{P}) = \sum_{k=1}^{n} m_k \left(\alpha(x_k) - \alpha(x_{k-1}) \right).$$

Since f is a bounded function, all M_k and m_k are finite. Hence, the upper and lower *Riemann-Stieltjes sums* are well-defined real numbers. Observe that if f is a nonnegative, continuous function on $[a, b]$ and $\alpha(x) = x$, then $RS^+(f,\alpha,\mathcal{P})$ and $RS^-(f,\alpha,\mathcal{P})$ are the sum of the areas of the circumscribed and inscribed rectangles induced by the partition $\mathcal{P}$ relative to the curve $y = f(x)$ over $[a, b]$.

Example 52

Let $f(x) = -8 + 6x - x^2$ on $[2, 5]$ and let $\mathcal{P} = \{2,\ 2.6,\ 3,\ 5\}$.

1. If $\alpha(x) = \sqrt{x}$, then

$$RS^+(f,\alpha,\mathcal{P}) = \sum_{k=1}^{n} M_k \left(\alpha(x_k) - \alpha(x_{k-1})\right)$$

$$= f(2.6)\ (\alpha(2.6) - \alpha(2)) + f(3)\ (\alpha(3) - \alpha(2.6) + f(3)\ (\alpha(5) - \alpha(3))$$

$$\approx 0.79013633716$$

and

$$RS^-(f,\alpha,\mathcal{P}) = \sum_{k=1}^{n} m_k \left(\alpha(x_k) - \alpha(x_{k-1})\right)$$

$$= f(0)\ (\alpha(2.6) - \alpha(2)) + f(2.6)\ (\alpha(3) - \alpha(2.6) + f(5)\ (\alpha(5) - \alpha(3))$$

$$\approx -11.803892979.$$

2. If

$$\alpha(x) = \begin{cases} 2 & \textit{if } x < 3.75 \\ x & \textit{if } x \geq 3.75 \end{cases},$$

then

$$RS^+(f,\alpha,\mathcal{P}) = \sum_{k=1}^{n} M_k \left(\alpha(x_k) - \alpha(x_{k-1})\right)$$

$$= f(2.6)\ (\alpha(2.6) - \alpha(2)) + f(3)\ (\alpha(3) - \alpha(2.6) + f(3)\ (\alpha(5) - \alpha(3))$$

$$\approx 3$$

and

$$
\begin{aligned}
RS^-(f,\alpha,\mathcal{P}) &= \sum_{k=1}^{n} m_k \left(\alpha(x_k) - \alpha(x_{k-1})\right) \\
&= f(0)\ (\alpha(2.6) - \alpha(2)) + f(2.6)\ (\alpha(3) - \alpha(2.6) + f(5)\ (\alpha(5) - \alpha(3)) \\
&\approx -9.
\end{aligned}
$$

Theorem 76

Suppose f maps $[a, b]$ into $[m, M]$ and α is nondecreasing on $[a, b]$.

(1) If $\mathcal{P}$ is a partition of $[a, b]$, then

$$m\left(\alpha(b) - \alpha(a)\right) \le RS^-(f,\alpha,\mathcal{P}) \le RS^+(f,\alpha,\mathcal{P}) \le M\left(\alpha(b) - \alpha(a)\right).$$

(2) If the partition Q is a refinement of the partition P on $[a, b]$, then

$$RS^-(f,\alpha,P) \le RS^-(f,\alpha,Q) \le RS^+(f,\alpha,Q) \le RS^+(f,\alpha,P).$$

(3) If P and Q are any two partitions of $[a, b]$, then

$$RS^-(f,\alpha,P) \le RS^+(f,\alpha,Q).$$

<u>Proof</u>

(1) Since $m \le m_k \le M_k \le M$ for $k = 1, 2, 3, \ldots, n$,

$$
\begin{aligned}
m\left(\alpha(b) - \alpha(a)\right) &= \sum_{k=1}^{n} m \left(\alpha(x_k) - \alpha(x_{k-1})\right) \qquad (\textit{telescopes}) \\
&\le \sum_{k=1}^{n} m_k \left(\alpha(x_k) - \alpha(x_{k-1})\right) \\
&\le \sum_{k=1}^{n} M_k \left(\alpha(x_k) - \alpha(x_{k-1})\right) \\
&\le \sum_{k=1}^{n} M \left(\alpha(x_k) - \alpha(x_{k-1})\right) \qquad (\textit{telescopes}) \\
&= M\left(\alpha(b) - \alpha(a)\right).
\end{aligned}
$$

The desired result is established.

(2) First, we suppose that Q contains exactly one more point than P. Let this point x^* be such that $x_{j-1} < x^* < x_j$ for some particular value of j. Then

$$
\begin{aligned}
RS^-(f,\alpha,P) &= \sum_{k=1}^{n} m_k \left(\alpha(x_k) - \alpha(x_{k-1})\right) \\
&= \sum_{k=1}^{j-1} m_k \left(\alpha(x_k) - \alpha(x_{k-1})\right) + m_j\left(\alpha(x_j) - \alpha(x_{j-1})\right) + \sum_{k=j+1}^{n} m_k\left(\alpha(x_k) - \alpha(x_{k-1})\right) \\
&= \sum_{k=1}^{j-1} m_k \left(\alpha(x_k) - \alpha(x_{k-1})\right) + m_j\left(\alpha(x^*) - \alpha(x_{j-1})\right) + \\
&\qquad m_j\left(\alpha(x_j) - \alpha(x^*)\right) + \sum_{k=j+1}^{n} m_k\left(\alpha(x_k) - \alpha(x_{k-1})\right) \\
&\le \sum_{k=1}^{j-1} m_k \left(\alpha(x_k) - \alpha(x_{k-1})\right) + \left(\inf_{x\in[x_{j-1},x^*]} f(x)\right)\left(\alpha(x^*) - \alpha(x_{j-1})\right) + \\
&\qquad \left(\inf_{x\in[x^*,x_j]} f(x)\right)\left(\alpha(x_j) - \alpha(x^*)\right) + \sum_{k=j+1}^{n} m_k\left(\alpha(x_k) - \alpha(x_{k-1})\right) \\
&= RS^-(f,\alpha,Q).
\end{aligned}
$$

By a mathematical induction argument on the number of extra points in Q, we have that

$$RS^-(f,\alpha,P) \le RS^-(f,\alpha,Q).$$

In a similar fashion, one can prove that

$$RS^+(f,\alpha,Q) \le RS^+(f,\alpha,P).$$

(3) Let T be a common refinement of the partitions P and Q. Then by parts (1) and (2), we have

$$RS^-(f,\alpha,P) \le RS^-(f,\alpha,T) \le RS^+(f,\alpha,T) \le RS^+(f,\alpha,Q). \ \blacktriangle$$

Definition 77

Let f be a bounded function on $[a,b]$ and let α a nondecreasing function on $[a,b]$. Then we define the ***upper Riemann-Stieltjes integral of f with respect to α over*** $[a,b]$ by

$$U\int_a^b f\,d\alpha = \underset{P}{glb}\ RS^+(f,\alpha,P)$$

and the ***lower Riemann-Stieltjes integral of f with respect to α over [a,b]*** by

$$L\int_a^b f\, d\alpha \;=\; \underset{P}{lub}\; RS^-(f,\alpha,P).$$

If $U\int_a^b f\, d\alpha \;=\; L\int_a^b f\, d\alpha$, then we say that f is ***Riemann-Stieltjes integrable with respect to α over*** $[a,b]$ and we denote this common value by

$$\int_a^b f\, d\alpha \quad \left(= \int_a^b f(x)\; d\alpha(x) \right).$$

We call the function f the ***integrand*** and the function α the ***integrator***. In the special case that $\alpha(x) = x$, we simply say that f is ***Riemann integrable***. By convention, if $a < b$, then we define

$$\int_b^a f\, d\alpha \;=\; -\int_a^b f\, d\alpha$$

and

$$\int_a^a f\, d\alpha \;=\; 0.$$

Theorem 78

Let f be a bounded function on $[a,b]$ and let α a nondecreasing function on $[a,b]$. Then

$$m\,(\alpha(b) - \alpha(a)) \le L\int_a^b f\, d \le U\int_a^b f\, d\alpha \le M\,(\alpha(b) - \alpha(a)).$$

Proof

We only prove that

$$L\int_a^b f\, d \le U\int_a^b f\, d\alpha.$$

By part (3) of Theorem 76, if P and Q are any two partitions of $[a,b]$, then

$$(*) \qquad RS^-(f,\alpha,P) \le RS^+(f,\alpha,Q).$$

Fix the partition P. Taking the greatest lower bound of the right hand side (*) as Q varies over all possible partitions Q of $[a, b]$ yields

$$(**) \qquad RS^-(f,\alpha,P) \leq \underset{Q}{glb}\ RS^+(f,\alpha,Q) = U\int_a^b f\, d\alpha .$$

Now, taking the least upper bound of the left hand side of (**) as P varies over all possible partitions of $[a, b]$ we obtain

$$L\int_a^b f\, d\alpha \leq U\int_a^b f\, d\alpha . \quad ▲$$

In the next example we show that both strict inequality and equality may occur.

<u>Example 53</u>

1. Let

$$f(x) = \begin{cases} 1 & if\ x \in \boldsymbol{Q} \\ 0 & if\ x \in \mathbb{R} - \boldsymbol{Q} \end{cases}$$

and let α be any nondecreasing function defined on $[a, b]$ with $\alpha(a) < \alpha(b)$. If P is any partition of $[a, b]$, then since $\boldsymbol{Q}$ & $\mathbb{R} - \boldsymbol{Q}$ are dense in $[a, b]$,

$$\begin{aligned} RS^-(f,\alpha,P) &= \sum_{k=1}^{n} m_k \left(\alpha(x_k) - \alpha(x_{k-1}) \right) \\ &= \sum_{k=1}^{n} 0 \left(\alpha(x_k) - \alpha(x_{k-1}) \right) \\ &\equiv 0. \end{aligned}$$

and

$$RS^+(f,\alpha,P) = \sum_{k=1}^{n} M_k \left(\alpha(x_k) - \alpha(x_{k-1}) \right)$$

$$= \sum_{k=1}^{n} 1 \left(\alpha(x_k) - \alpha(x_{k-1}) \right) \qquad (\textit{telescopes})$$

$$\equiv \alpha(x_n) - \alpha(x_0)$$

$$> 0.$$

Hence,

$$L\int_a^b f\, d\alpha = 0 < \alpha(b) - \alpha(a) = U\int_a^b f\, d\alpha.$$

We conclude that the ***Dirichlet function*** is not Riemann-Stieltjes integrable with respect to any nonconstant integrator function α. In particular, *the Dirichlet function is not Riemann integrable.*

2. Let $t \in (a, b)$ and let $c_1 < c_2$. Suppose that f is a bounded function on $[a, b]$ that is continuous at $x = t$ and suppose that

$$\alpha(x) = \begin{cases} c_1 & \textit{if } x < t \\ c_2 & \textit{if } x \geq t. \end{cases}$$

Let P be any partition of $[a, b]$. Then

$$RS^-(f,\alpha,P) \leq L\int_a^b f\, d\alpha \leq U\int_a^b f\, d\alpha \leq RS^+(f,\alpha,P).$$

Let $\varepsilon > 0$ be given. Since f is continuous at $x = t$, there exists a $\delta > 0$ so that if

$$|x - t| < \delta,$$

we have

$$|f(x) - f(t)| < \varepsilon \quad \Leftrightarrow \quad f(t) - \varepsilon < f(x) < f(t) + \varepsilon.$$

Choose a partition $P = \{ x_0 = a,\ x_1,\ x_2,\ x_3 = b \}$ where $t - \delta < x_1 < t < x_2 < t + \delta$. Then

$$
\begin{aligned}
RS^-(f,\alpha,P) &= m_2\left(\alpha(x_2) - \alpha(x_1)\right)\\
&= m_2\left(c_2 - c_1\right)\\
&\geq \left(f(t) - \varepsilon\right)\left(c_2 - c_1\right)\\
&= f(t)\left(c_2 - c_1\right) - \varepsilon\left(c_2 - c_1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
RS^+(f,\alpha,P) &= M_2\left(\alpha(x_2) - \alpha(x_1)\right)\\
&= M_2\left(c_2 - c_1\right)\\
&\leq \left(f(t) + \varepsilon\right)\left(c_2 - c_1\right)\\
&= f(t)\left(c_2 - c_1\right) + \varepsilon\left(c_2 - c_1\right).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
f(t)\,(c_2 - c_1) - \varepsilon\,(c_2 - c_1) &\leq L\int_a^b f\,d\alpha\\
&\leq U\int_a^b f\,d\alpha\\
&\leq f(t)\,(c_2 - c_1) + \varepsilon\,(c_2 - c_1).
\end{aligned}
$$

The above holds for all $\varepsilon > 0$. Taking the limit of the above as $\varepsilon \to 0$ produces

$$\int_a^b f\,d\alpha = f(t)\left(c_2 - c_1\right)$$

or

$$\int_a^b f\,d\alpha = f(t)\left(\alpha(t^+) - \alpha(t^-)\right).$$

That is, f is Riemann-Stieltjes integrable with respect to α.

3. Let $a < t_1 < t_2 < ... < t_n < b$. Suppose that f is bounded on $[a,b]$ and continuous at least at the points t_k, $k = 1, 2, 3, \ldots, n$. Let $c_0 < c_1 < c_2 < ... < c_n$ and define

$$\alpha(x) = \begin{cases} c_0 & \text{if } x \in [a,t_1) \\ c_1 & \text{if } x \in [t_1,t_2) \\ c_2 & \text{if } x \in [t_2,t_3) \\ & \dots \\ c_{n-1} & \text{if } x \in [t_{n-1},t_n) \\ c_n & \text{if } x \in [t_n,b] \end{cases}$$

Generalizing part 2, one can show that f is Riemann-Stieltjes integrable with respect to α and that

$$\int_a^b f\, d\alpha = f(t_1)(c_1 - c_0) + f(t_2)(c_2 - c_1) + \dots + f(t_n)(c_n - c_{n-1})$$

or

$$\int_a^b f\, d\alpha = \sum_{k=1}^{n} f(t_k)\left(\alpha(t_k^+) - \alpha(t_k^-)\right).$$

Note: A necessary condition for the existence of $\int_a^b f\, d\alpha$ is that f and α not share a point of discontinuity.

Theorem 79 - Riemann's Condition for Integrability

Suppose that f is bounded on $[a,b]$ and α is a nondecreasing function on $[a,b]$. Then f is Riemann-Stieltjes integrable with respect to α on $[a,b]$ iff for each $\varepsilon > 0$ there exists a partition P of $[a,b]$ (depending on ε) so that

$$RS^+(f,\alpha,P) - RS^-(f,\alpha,P) < \varepsilon.$$

Proof

($\Rightarrow$) Assume that f is Riemann-Stieltjes integrable with respect to α on $[a,b]$. Let $\varepsilon > 0$ be given. Since

$$L\int_a^b f\, d\alpha = \underset{\wp}{lub}\ RS^-(f,\alpha,\wp)$$

where $\wp$ ranges over all possible partitions of $[a, b]$, there exists a partition P_1 of $[a, b]$ such that

(i) $$L\int_a^b f\,d\alpha - \frac{\varepsilon}{2} < RS^-(f,\alpha,P_1) \le L\int_a^b f\,d\alpha.$$

Because

$$U\int_a^b f\,d\alpha = \underset{\wp}{glb}\ RS^+(f,\alpha,\wp)$$

where $\wp$ ranges over all possible partitions of $[a, b]$, there exists a partition P_2 of $[a, b]$ so that

(ii) $$U\int_a^b f\,d\alpha \le RS^+(f,\alpha,P_2) < U\int_a^b f\,d\alpha + \frac{\varepsilon}{2}.$$

Let P_3 be a common refinement of P_1 and P_2. By Theorem 76 and the inequalities (i) and (ii) we have that

$$L\int_a^b f\,d\alpha - \frac{\varepsilon}{2} < RS^-(f,\alpha,P_3) \le RS^+(f,\alpha,P_3) < U\int_a^b f\,d\alpha + \frac{\varepsilon}{2}.$$

It follows that

(iii) $$\begin{aligned} RS^+(f,\alpha,P_3) - RS^-(f,\alpha,P_3) &< \left(U\int_a^b f\,d\alpha + \frac{\varepsilon}{2} \right) - \left(L\int_a^b f\,d\alpha - \frac{\varepsilon}{2} \right) \\ &= \left(U\int_a^b f\,d\alpha - L\int_a^b f\,d\alpha \right) + \varepsilon. \end{aligned}$$

Since f is Riemann-Stieltjes integrable with respect to α on $[a, b]$,

$$L\int_a^b f\,d\alpha = U\int_a^b f\,d\alpha.$$

Thus, (iii) becomes

$$RS^+(f,\alpha,P_3) - RS^-(f,\alpha,P_3) < \varepsilon.$$

($\Leftarrow$) Suppose that for each $\varepsilon > 0$ there exists a partition P of $[a, b]$ so that

(iv) $$RS^+(f,\alpha,P) - RS^-(f,\alpha,P) < \varepsilon.$$

Let $\varepsilon > 0$ be given and let P be the partition of $[a, b]$ corresponding to ε so that (iv) is satisfied.

By Definition 77,

$$RS^-(f,\alpha,P) \le L\int_a^b f\,d\alpha \le U\int_a^b f\,d\alpha \le RS^+(f,\alpha,P)$$

and so we have

$$U\int_a^b f\,d\alpha - L\int_a^b f\,d\alpha \le RS^+(f,\alpha,P) - RS^-(f,\alpha,P) < \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, we conclude that

$$L\int_a^b f\,d\alpha = U\int_a^b f\,d\alpha.$$

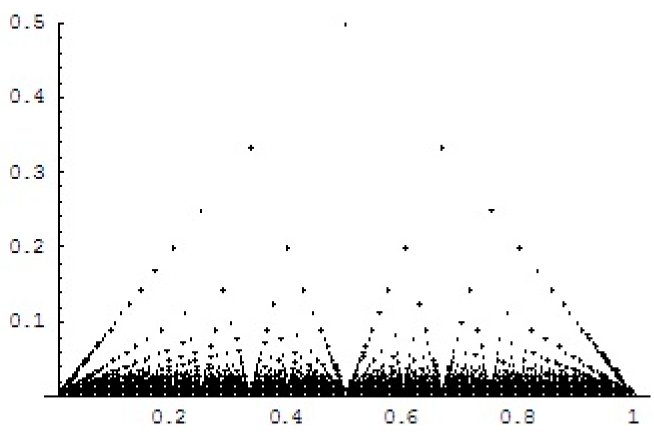
Hence, f is Riemann-Stieltjes integrable with respect to α on $[a, b]$. ▲

Example 54

Suppose that $f:[0,1] \to \mathbb{R}$ is defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ is rational} \end{cases}$$

where $\dfrac{p}{q}$ is a fraction in lowest terms. The graph of $y = f(x)$ over [0,1] is shown below:



The function $y = f(x)$ is continuous at each irrational x and discontinuous at each rational x. If

$\alpha(x) = x$ on $[0,1]$, then it can be shown that f is RS-integrable w/r/t α and that $\int_0^1 f\,d\alpha = 0$.

Theorem 80

If f is continuous on $[a,b]$ and α is a nondecreasing function on $[a,b]$ with $\alpha(b) - \alpha(a) > 0$, then f is Riemann-Stieltjes integrable with respect to α on $[a,b]$.

Proof

We use ***Riemann's Condition for Integrability*** to establish the result. Let $\varepsilon > 0$ be given. Since is f is continuous on the closed interval $[a,b]$, f is uniformly continuous there. So, there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{\alpha(b) - \alpha(a)}$$

whenever

$$|x - y| < \delta.$$

Let $P = \{ x_0, x_1, x_2, \ldots, x_n \}$ be a partition of $[a,b]$ with $x_k - x_{k-1} < \delta$. Then

$$\begin{aligned}
RS^+(f,\alpha,P) - RS^-(f,\alpha,P) &= \sum_{k=1}^{n} (M_k - m_k)\,(\alpha(x_k) - \alpha(x_{k-1})) \\
&= \sum_{k=1}^{n} (f(s_k) - f(t_k))\,(\alpha(x_k) - \alpha(x_{k-1})) \quad \left(s_k,\ t_k \in [x_{k-1}, x_k]\right) \\
&< \sum_{k=1}^{n} \frac{\varepsilon}{\alpha(b) - \alpha(a)}\,(\alpha(x_k) - \alpha(x_{k-1})) \\
&= \frac{\varepsilon}{\alpha(b) - \alpha(a)} \sum_{k=1}^{n} (\alpha(x_k) - \alpha(x_{k-1}) \quad (\textit{telescopes}) \\
&= \frac{\varepsilon}{\alpha(b) - \alpha(a)}\,(\alpha(b) - \alpha(a)) \\
&= \varepsilon
\end{aligned}$$

where we used the **Extreme Value Theorem** to obtain $M_k = f(s_k)$ and $m_k = f(t_k)$ for some $s_k, t_k \in [x_{k-1}, x_k]$. By ***Riemann's Condition for Integrability***, f is Riemann-Stieltjes integrable with respect to α on $[a, b]$. ▲

Theorem 81

If f is a monotone function on $[a, b]$ and if α is a continuous, nondecreasing function on $[a, b]$ with $\alpha(b) - \alpha(a) > 0$, then f is Riemann-Stieltjes integrable with respect to α on $[a, b]$.

Proof (Optional)

We use ***Riemann's Condition for Integrability*** to establish the result. We assume that f is increasing on $[a, b]$. (The decreasing case is done a similar manner.) Let $\varepsilon > 0$ be given. Find a natural number N so that

$$\frac{\alpha(b) - \alpha(a)}{N}\left(f(b) - f(a)\right) < \varepsilon.$$

Since α is continuous on $[a, b]$, we repeatedly apply the ***Intermediate Value Theorem*** to find $x_k \in [a,b]$, $k = 1, 2, 3, \ldots, N-1$, so that

$$\alpha\left(x_k\right) = \alpha(a) + k\,\frac{\alpha(b) - \alpha(a)}{N}.$$

Then $P = \{ x_0 = a, x_1, x_2, \ldots , x_{N-1}, x_N = b \}$ is a partition of $[a, b]$ with

$$\alpha(x_k) - \alpha(x_{k-1}) = \frac{\alpha(b) - \alpha(a)}{N}.$$

Hence,

$$
\begin{aligned}
RS^{+}(f,\alpha,P) - RS^{-}(f,\alpha,P) &= \sum_{k=1}^{N} (M_k - m_k)\,(\alpha(x_k) - \alpha(x_{k-1})) \\
&= \sum_{k=1}^{N} \left(f(x_k) - f(x_{k-1})\right)\left(\alpha(x_k) - \alpha(x_{k-1})\right) \qquad (f\uparrow) \\
&= \sum_{k=1}^{N} \left(f(x_k) - f(x_{k-1})\right) \frac{\alpha(b) - \alpha(a)}{N} \\
&= \frac{\alpha(b) - \alpha(a)}{N} \sum_{k=1}^{N} \left(f(x_k) - f(x_{k-1})\right) \quad (\textit{telescopes}) \\
&= \frac{\alpha(b) - \alpha(a)}{N} \left(f(b) - f(a)\right) \\
&< \varepsilon.
\end{aligned}
$$

Thus, by ***Riemann's Condition for Integrability***, f is Riemann-Stieltjes integrable w/r/t α on $[a, b]$. ▲

The previous two theorems provide us with a wealth of functions that are Riemann-Stieltjes integrable although neither result gives us a means to evaluate $\int_a^b f\, d\alpha$ *beyond that given in Definition 77!* At this point actually calculating the value of the RS-integral is still a highly nontrivial task as the next example illustrates.

<u>Example 55</u>

Let $f(t) = t^3$ & $\alpha(t) = t^2$ on $[1, 5]$ and let

$$P_n = \left\{x_0 = 1, x_1 = 1 + \frac{4}{n}, x_2 = 1 + \frac{4(2)}{n}, x_3 = 1 + \frac{4(3)}{n}, \ldots, x_{n-1} = 1 + \frac{4(n-1)}{n}, x_n = 5\right\}$$

be a regular partition of $[1, 5]$. By the above theorems we have that f is RS-integrable w/r/t α on $[1, 5]$. Now,

$$RS^{+}\left(f,\alpha,P_n\right) = \sum_{k=1}^{n} M_k \left(\alpha\left(x_k\right) - \alpha\left(x_{k-1}\right)\right)$$

$$= \sum_{k=1}^{n} \left(1 + \frac{4k}{n}\right)^3 \left(\left(1 + \frac{4k}{n}\right)^2 - \left(1 + \frac{4(k-1)}{n}\right)^2\right)$$

$$= \sum_{k=1}^{n} \left\{\left(\frac{2048}{n^5}\right) k^4 + \left(\frac{2048}{n^4} - \frac{1024}{n^5}\right) k^3 \right.$$

$$\left. + \left(\frac{768}{n^3} - \frac{768}{n^4}\right) k^2 + \left(\frac{128}{n^2} - \frac{192}{n^2}\right) k + \left(\frac{8}{n} - \frac{16}{n^2}\right)\right\}$$

$$= \left(\frac{2048}{n^5}\right) \sum_{k=1}^{n} k^4 + \left(\frac{2048}{n^4} - \frac{1024}{n^5}\right) \sum_{k=1}^{n} k^3$$

$$+ \left(\frac{768}{n^3} - \frac{768}{n^4}\right) \sum_{k=1}^{n} k^2 + \left(\frac{128}{n^2} - \frac{192}{n^2}\right) \sum_{k=1}^{n} k + \left(\frac{8}{n} - \frac{16}{n^2}\right) \sum_{k=1}^{n} 1$$

$$= \left(\frac{2048}{n^5}\right) \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} + \left(\frac{2048}{n^4} - \frac{1024}{n^5}\right) \frac{n^2(n+1)^2}{4}$$

$$+ \left(\frac{768}{n^3}\right) \frac{n(n+1)(2n+1)}{6} + \left(\frac{128}{n^2} - \frac{192}{n^2}\right) \frac{n(n+1)}{2} + \left(\frac{8}{n} - \frac{16}{n^2}\right) n$$

$$= \frac{8\left(2343\, n^4 + 3510\, n^3 + 620\, n^2 - 720\, n - 128\right)}{15\, n^4}$$

and taking the limit of the above as $n \to \infty$ yields $\frac{6248}{5}$. That is, $RS^{+}(f,\alpha,P_n)$ is a sequence of real numbers converging to $\frac{6248}{5}$. Similarly, $RS^{-}(f,\alpha,P_n) \to \frac{6248}{5}$ as $n \to \infty$. We can conclude that

$$\int_1^5 f\, d\alpha = \frac{6248}{5}.$$

The next three results gives some of the standard properties for the Riemann-Stieltjes integral.

Theorem 82

(1) If f and g are both Riemann-Stieltjes integrable with respect to α on $[a, b]$ and if c and d are any two real numbers, then $cf + dg$ is Riemann-Stieltjes integrable w/r/t to α on $[a, b]$ and

$$\int_a^b (c\,f + d\,g)\,d\alpha = c\int_a^b f\,d\alpha + d\int_a^b g\,d\alpha.$$

(2) If f is Riemann-Stieltjes integrable with respect to both α and β on $[a, b]$ and if c and d are nonnegative real numbers, then f is Riemann-Stieltjes integrable with respect to $\gamma = c\,\alpha + d\,\beta$ and

$$\int_a^b f\,d\gamma = c\int_a^b f\,d\alpha + d\int_a^b f\,d\beta.$$

(3) If f and g are both Riemann-Stieltjes integrable with respect to α on $[a, b]$, then fg is Riemann-Stieltjes integrable with respect to α on $[a, b]$.

Part (1) of Theorem 82 implies that the collection of all Riemann-Stieltjes integrable functions with respect to a fixed nondecreasing function α form a vector space over the real numbers. Further, the function T defined by

$$T(f) = \int_a^b f\,d\alpha$$

is a ***linear transformation*** (in the sense of linear algebra!) from the collection of all Riemann-Stieltjes integrable functions with respect to a fixed nondecreasing function α into the vector space $\mathbb{R}^1$.

Theorem 83

(1) If f is Riemann-Stieltjes integrable with respect to α on $[a, c]$ and $[c, b]$, then f is Riemann-Stieltjes integrable with respect to α on $[a, b]$ and

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

(2) If f is Riemann-Stieltjes integrable with respect to α on $[a,b]$ and if $a < c < b$, then f is Riemann-Stieltjes integrable with respect to α on $[a,c]$ and $[c,b]$ and

$$\int_a^b f\,d\alpha = \int_a^c f\,d\alpha + \int_c^b f\,d\alpha.$$

Proof

(1) Let $\varepsilon > 0$ be given. By ***Riemann's Condition for Integrability***, there exist partitions P and Q of $[a,c]$ and $[c,b]$ respectively so that

$$RS^{+}(f,\alpha,P) - RS^{-}(f,\alpha,P) < \frac{\varepsilon}{2}$$

and

$$RS^{+}(f,\alpha,Q) - RS^{-}(f,\alpha,Q) < \frac{\varepsilon}{2}.$$

Let $R = P \cup Q$. Then R is a partition of $[a,b]$ with

$$\begin{aligned} RS^{+}(f,\alpha,R) - RS^{-}(f,\alpha,R) &= \left(RS^{+}(f,\alpha,P) + RS^{+}(f,\alpha,Q)\right) - \left(RS^{-}(f,\alpha,P) + RS^{-}(f,\alpha,Q)\right) \\ &= \left(RS^{+}(f,\alpha,P) - RS^{-}(f,\alpha,P)\right) + \left(RS^{+}(f,\alpha,Q) + RS^{-}(f,\alpha,Q)\right) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, f is Riemann-Stieltjes integrable w/r/t α on $[a,b]$.

Since

$$\begin{aligned} \int_a^c f\,d\alpha + \int_c^b f\,d\alpha &\le RS^{+}(f,\alpha,P) + RS^{+}(f,\alpha,Q) \\ &= RS^{+}(f,\alpha,P \cup Q) \end{aligned},$$

for all partitions P of $[a, c]$ and all partition Q of $[c, b]$, it follows that

$$\int_a^c f\, d\alpha + \int_c^b f\, d\alpha \le U\!\int_a^b f\, d\alpha .$$

In a similar manner, we have that

$$L\!\int_a^b f\, d\alpha \le \int_a^c f\, d\alpha + \int_c^b f\, d\alpha .$$

Because f is Riemann-Stieltjes integrable w/r/t α on $[a, b]$,

$$L\!\int_a^b f\, d\alpha = \int_a^b f\, d\alpha = U\!\int_a^b f\, d\alpha$$

and so

$$\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha . \quad \blacktriangle$$

Theorem 84

If f is Riemann-Stieltjes integrable with respect to α on $[a, b]$, then $|f|$ is Riemann-Stieltjes integrable with respect to α on $[a, b]$ and

$$\left| \int_a^b f\, d\alpha \right| \le \int_a^b |f|\, d\alpha .$$

The next result relates the Riemann integral of ordinary calculus to the Riemann-Stieltjes integral. One consequence of the result is that we will be able to evaluate many Riemann-Stieltjes integrals by transforming them into Riemann integrals and applying the methods of integration from ordinary calculus.

Theorem 85

Suppose that f is a bounded function on $[a, b]$ and α is a nondecreasing function on $[a, b]$ with α' Riemann integrable on $[a, b]$. Then f is Riemann-Stieltjes integrable with respect to α on $[a, b]$ iff $f\alpha'$ is Riemann integrable on $[a, b]$. In this case,

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\, \alpha'(x)\, dx .$$

Proof

Let $\varepsilon > 0$ be given. The result is trivial if f vanishes on [a,b]. So, suppose that

$$M = \sup_{x \in [a,b]} |f(x)| > 0.$$

Since α' is Riemann integrable on [a,b], we apply Theorem 79 to obtain a partition $P = \{x_0, x_1, \ldots, x_n\}$ of [a,b] so that

$$(*) \qquad RS^{+}(\alpha', x, P) - RS^{-}(\alpha', x, P) < \frac{\varepsilon}{M}.$$

Because α is continuous on [a,b] and differentiable on (a,b), we apply the *Mean Value Theorem* to each subinterval $[x_{k-1}, x_k]$ obtaining a point $t_k \in [x_{k-1}, x_k]$ with

$$\alpha(x_k) - \alpha(x_{k-1}) = \alpha'(t_k)\,(x_k - x_{k-1}).$$

For $k = 1, \ldots, n$ let $s_k \in [x_{k-1}, x_k]$. Then

$$\sum_{k=1}^{n} |\alpha'(s_k) - \alpha'(t_k)|\left(x_k - x_{k-1}\right) \le RS^{+}(\alpha', x, P) - RS^{-}(\alpha', x, P) < \frac{\varepsilon}{M}.$$

So, for any $s_k \in [x_{k-1}, x_k]$ we have

$$\left|\sum_{k=1}^{n} f(s_k)\left(\alpha(x_k) - \alpha(x_{k-1})\right) - \sum_{k=1}^{n} f(s_k)\,\alpha'(s_k)\left(x_k - x_{k-1}\right)\right|$$

$$\begin{aligned}
&= \left|\sum_{k=1}^{n} f(s_k)\,\alpha'(t_k)\,(x_k - x_{k-1}) - \sum_{k=1}^{n} f(s_k)\,\alpha'(s_k)(x_k - x_{k-1})\right| \\
&\le \sum_{k=1}^{n} \left|f(s_k)\left(\alpha'(t_k) - \alpha'(s_k)\right)\left(x_k - x_{k-1}\right)\right| \\
&\le M \sum_{k=1}^{n} \left|\alpha'(t_k) - \alpha'(s_k)\right|\left(x_k - x_{k-1}\right) \\
&< M\,\frac{\varepsilon}{M} \\
&= \varepsilon.
\end{aligned}$$

Since the points $s_k \in [x_{k-1}, x_k]$ were arbitrary,

$$\sum_{k=1}^{n} f(s_k) \left(\alpha(x_k) - \alpha(x_{k-1})\right) \leq \sum_{k=1}^{n} f(s_k)\ \alpha'(s_k)\ (x_k - x_{k-1}) + \varepsilon$$
$$\leq RS^+(f\alpha', x, P) + \varepsilon.$$

Thus, $RS^+(f\alpha', x, P) + \varepsilon$ is an upper bound for all sums of the form $\sum_{k=1}^{n} f(s_k) \left(\alpha(x_k) - \alpha(x_{k-1})\right)$ and

so we have that

$$RS^+(f, \alpha, P) \leq RS^+(f\alpha', x, P) + \varepsilon.$$

In a similar fashion,

$$RS^+(f\alpha', x, P) \leq RS^+(f, \alpha, P) + \varepsilon.$$

Hence,

$$(**) \qquad \left|RS^+(f, \alpha, P) - RS^+(f\alpha', x, P)\right| < \varepsilon.$$

Since (*) holds for any refinement of P, it follows that (**) does as well. From this it follows that

$$\left|U\int_a^b f\, d\alpha - U\int_a^b f\alpha'\right| < \varepsilon..$$

Because $\varepsilon > 0$ was arbitrary, we conclude that

$$U\int_a^b f\, d\alpha = U\int_a^b f\alpha'.$$

A similar argument shows that

$$L\int_a^b f\, d\alpha = L\int_a^b f\alpha'.$$

Thus, f is Riemann-Stieltjes integrable with respect to α on [a,b] iff $f\alpha'$ is Riemann integrable on [a,b]. In this case, the above shows that

$$\int_a^b f\, d\alpha = \int_a^b f\alpha'. \quad \blacktriangle$$

Example 56

Let $f(t) = t^3$ & $\alpha(t) = t^2$ on $[1, 5]$. By Example 55 we know that $\int_1^5 f\, d\alpha = \frac{6248}{5}$.

Applying the above theorem we have

$$\int_1^5 t^3\, d(t^2) = \int_1^5 t^3\, (t^2)\, dt$$

$$= \int_1^5 2\,t^4\, dt$$

$$= \frac{2\,t^5}{5}\Bigg|_1^5$$

$$= \frac{6248}{5}.$$

Example 57

(1) $$\int_0^1 x^2\, d(x^3) = \int_0^1 x^2\, (3x^2)\, dx$$
$$= \int_0^1 3x^4\, dx$$
$$= \frac{3}{5}\, x^5 \,\Big|_0^1$$
$$= \frac{3}{5}$$

(2) $$\int_0^{\pi/2} x\, d(\sin x) = \int_0^{\pi/2} x\,(-\cos x)\, dx$$
$$= -\cos x - x \sin x \,\Big|_0^{\pi/2} \quad (\textit{Integration by Parts})$$
$$= 1 - \frac{\pi}{2}.$$

Note: $\alpha(x) = \sin x$ is increasing on the interval $[0, \pi/2]$.

(3) $\int_0^3 [\![x]\!] \, d(x^2) = \int_1^2 (1)\,(2x)\, dx + \int_2^3 (2)\,(2x)\, dx$

$$= x^2 \Big|_1^2 + 2x^2 \Big|_2^3$$

$$= 3 + 10$$

$$= 13.$$

The next result combines the ideas of Example 53 and Theorem 85 to yield an extremely useful tool in the evaluation of Riemann-Stieltjes integrals.

Theorem 86

Suppose that f is a bounded function on $[a,b]$ and α is a nondecreasing function on $[a,b]$. If for some c in (a,b) we that

(i) $f\alpha'$ has an antiderivative $F(x)$ on $[a,c)$ and $\lim\limits_{x \to c^-} F(x) = F(c^-)$ exists;

(ii) $f\alpha'$ has an antiderivative $G(x)$ on $(c,b]$ and $\lim\limits_{x \to c^+} G(x) = G(c^+)$ exists;

(iii) α has a jump discontinuity at c;

(iv) f is continuous at c,

then

$$\int_a^b f \, d\alpha = \big(F(c^-) - F(a)\big) + f(c)\,\big(\alpha(c^+) - \alpha(c^-)\big) + \big(G(b) - G(c^+)\big).$$

The above result remains valid if α has a finite number of discontinuities and f is continuous at the points where α is discontinuous.

__**Example 58**__

(1) Let $f(x) = e^x$ and

$$\alpha(x) = \begin{cases} x^2 & \text{if } 0 \le x < 1 \\ 1 + 2x & \text{if } 1 \le x \le 2 \\ 5 + x & \text{if } 2 < x \le 3 \end{cases}$$

Then f is continuous on $[0,3]$ and α has jump discontinuities at $x = 1$ and $x = 2$. So, by Theorem 86,

$$\int_0^3 f\, d\alpha = \int_0^1 e^x (2x)\, dx + e^1 (\alpha(1^+) - \alpha(1^-)) + \int_1^2 e^x (2)\, dx + e^2 (\alpha(2^+) - \alpha(2^-)) + \int_2^3 e^x (1)\, dx$$

$$= \left[2\left(xe^x \Big|_0^1 - \int_0^1 e^x\, dx \right) \right] + e(3-1) + 2(e^2 - e^1) + e^2(7-5) + (e^3 - e^2)$$

$$= 2 + 3e^2 + e^3.$$

(2) Let $f(x) = x^2 + 3$ and

$$\alpha(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 + x^3 & \text{if } 0 < x \leq 1 \end{cases}$$

Then

$$\int_0^1 f\, d\alpha = f(0)\left(\alpha(0^+) - \alpha(0)\right) + \int_0^1 f(x)\, \alpha'(x)\, dx$$

$$= 3\ (1 - 0) + \int_0^1 \left(3x^4 + 9x^2\right) dx$$

$$= 3 + \left(\frac{3}{5} x^5 + 3x^3 \Big|_0^1 \right)$$

$$= \frac{33}{5}.$$

One of the standard techniques of integration in ordinary calculus was the method of *Integration by Parts*. This technique remains valid for the Riemann-Stieltjes integral when the functions involved are monotonic.

Theorem 87 - Integration by Parts

Suppose f and g are nondecreasing on $[a,b]$. Then f is Riemann-Stieltjes integrable with respect to g on $[a,b]$ iff g is Riemann-Stieltjes integrable with respect to f on [a,b]. In this case, we have

$$\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - \int_a^b g\,df.$$

The proof of the *Integration by Parts* formula utilizes the following lemma.

Lemma

If f and g are two nondecreasing functions on $[a,b]$ and P is any partition of $[a,b]$, then

$$RS^+(f,g,P) = f(b)g(b)\ f(a)g(a) - RS^-(g,f,P)$$

and

$$RS^-(f,g,P) = f(b)g(b)\ f(a)g(a) - RS^+(g,f,P).$$

We omit the proof of the lemma (See Kirkwood pp169-70.)

Proof of Theorem 87

Assume that f is Riemann-Stieltjes integrable with respect to g on $[a,b]$. Let $\varepsilon > 0$ be given. Then, by Riemann's Condition, there exists a partition P of $[a,b]$ so that

$$RS^+(f,g,P) - RS^-(f,g,P) < \varepsilon.$$

By the lemma,

$$\begin{aligned} RS^+(f,g,P) - RS^-(f,g,P) &= \big(f(b)g(b) - f(a)g(a) - RS^-(g,f,P)\big) \\ &\quad - \big(f(b)g(b) - f(a)g(a) - RS^+(g,f,P)\big) \\ &= RS^+(g,f,P) - RS^-(g,f,P). \end{aligned}$$

It follows from Riemann's Condition that g is Riemann-Stieltjes integrable with respect to f on $[a,b]$.

Thus, f and g are Riemann-Stieltjes integrable w/r/t each other and there are partitions P_1 and P_2 so that

$$RS^+(f,g,P_1) - \int_a^b f\,dg < \frac{\varepsilon}{2}$$

and

$$\int_a^b g\,df - RS^-(g,f,P_2) < \frac{\varepsilon}{2}.$$

Let P_3 be a common refinement of P_1 and P_2. Then

$$\left|\int_a^b f\,dg - \left(f(b)g(b) - f(a)g(a) - \int_a^b g\,df\right)\right|$$

$$= \left|\left(\int_a^b f\,dg - RS^+(f,g,P_3)\right) + \left\{RS^+(f,g,P_3) - \left[f(b)g(b) - f(a)g(a) - \int_a^b g\,df\right]\right\}\right|$$

$$= \left|\left(\int_a^b f\,dg - RS^+(f,g,P_3)\right) + \left[f(b)g(b) - f(a)g(a) - RS^-(g,f,P_3)\right] - \left[f(b)g(b) - f(a)g(a) - \int_a^b g\,df\right]\right|$$

$$= \left|\int_a^b f\,dg - RS^+(f,g,P_3)\right| + \left|\int_a^b g\,df - RS^-(g,f,P_3)\right|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have that

$$\int_a^b f\,dg = f(b)g(b) - f(a)g(a) - \int_a^b g\,df. \quad ▲$$

Example 59

Apply the *Integration by Parts* formula (Theorem 87) to show that if f is a continuous, increasing function on $[a,b]$, then

$$\int_a^b f\,df = \frac{f(b)^2 - f(a)^2}{2}.$$

We now use the *Intermediate Value Theorem* to prove the *Mean Value Theorem for Integrals*.

Theorem 88 - Mean Value Theorem for Integrals

Suppose that f is continuous on $[a,b]$ and α is increasing on $[a,b]$. Then there exists a number c with $a < c < b$ so that

$$\int_a^b f\,d\alpha = f(c)\int_a^b d\alpha = f(c)\,(\alpha(b) - \alpha(a)).$$

Proof

By the *Extreme Value Theorem*, f assumes its maximum and minimum values on $[a,b]$, say $M = \max_{x\in[a,b]} f(x)$ and $m = \min_{x\in[a,b]} f(x)$. It follows that

$$(*) \qquad m \le \frac{1}{\alpha(b) - \alpha(a)}\int_a^b f\,d\alpha \le M.$$

Because f is continuous on $[a,b]$ and (*) shows that the real number given by

$$\frac{1}{\alpha(b) - \alpha(a)}\int_a^b f\,d\alpha$$

is an intermediate value for f, there exists a number c between a and b so that

$$f(c) = \frac{1}{\alpha(b) - \alpha(a)}\int_a^b f\,d\alpha.$$

We now have the desired result. ▲

Theorem 89 - Differentiation Theorem

Suppose that f is continuous on $[a,b]$ and α is an increasing function on $[a,b]$ that is differentiable at c in (a,b). Then the function $F:[a,b]\to\mathbb{R}$ defined by

$$F(x) = \int_a^x f\,d\alpha$$

is differentiable at c and $F'(c) = f(c)\,\alpha'(c)$.

Proof

We show that F has a right-hand derivative at $x = c$. Let $h > 0$ be choosen so that $c+h \in (a,b)$. Then

$$\frac{F(c+h) - F(c)}{h} = \frac{1}{h}\left[\int_a^{c+h} f\, d\alpha - \int_a^{c} f\, d\alpha\right]$$

$$= \frac{1}{h}\left[\int_c^{c+h} f\, d\alpha\right]$$

$$= f(c^*)\ \frac{\alpha(c+h) - \alpha(c)}{h} \quad (\textit{MVT for Integrals}).$$

where $c^* \in (c,c+h)$. Since f is continuous at c and α' is differentiable at c,

$$\lim_{h\to 0^+} \frac{F(c+h) - F(c)}{h} = f(c)\ \alpha'(c).$$

In a similar fashion,

$$\lim_{h\to 0^-} \frac{F(c+h) - F(c)}{h} = f(c)\ \alpha'(c).$$

Hence, F is differentiable at c and $F'(c) = f(c)\ \alpha'(c)$. ▲

Setting $\alpha(x) = x$ in the above produces the ***Fundamental Theorem of Calculus*** of ordinary calculus.

We now consider $\int_a^b f\, d\alpha = \lim_{n\to\infty} \int_a^b f_n\, d\alpha$ where $f_n \to f$ in some sense on $[a,b]$.

First a little background material:

Definition 90

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on $[a,b]$ and let f be a function whose domain contains $[a,b]$.

(1) We say that f_n ***converges pointwise on* [*a,b*] *to*** f, denoted $f_n \to f$, iff for every $x \in [a,b]$ we have that $\lim_{n\to\infty} f_n(x) = f(x)$.

(2) We say that f_n ***converges uniformly on* [*a,b*] *to*** f, denoted by $f_n \underset{unif}{\to} f$, iff for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that

$$\left| f_n(x) - f(x) \right| < \varepsilon$$

for all $x \in [a,b]$.

Example 60

(1) Suppose that $f_n(x) = x^n$ ($n = 1, 2, 3, \ldots$) on $[0,1]$. It follows that

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}.$$

So, $f_n \to f$ on $[0,1]$. By choosing $0 < \varepsilon < 1$ we see that f_n fails to converge uniformly to f on $[0,1]$. Observe that each function f_n is continuous on [0,1] yet the pointwise limit f fails to be continuous there. Hence, pointwise converges fails to preserve continuity.

(2) Let $\{x_n\}$ be an enumeration of the rational numbers in $[0,1]$. (Since the set of rational numbers is countable in $[0,1]$, there exists a bijection $r : \mathbb{N} \to \boldsymbol{Q} \cap [0,1]$ with $r(n) = x_n$. We call r an ***enumeration*** of the rationals in [0,1].) Define

$$f_n(x) = \begin{cases} 0 & \text{if } x = x_k,\ 1 \le k \le n \\ 1 & \text{otherwise} \end{cases}.$$

Then each f_n is continuous and equals 1 on [0,1] except for a finite number of discontinuities at $x = x_1, x_2, x_3, \ldots, x_n$ and

$$\int_0^1 f_n(x)\, dx = 1.$$

Here

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}.$$

Again, $f_n \to f$ on [0,1] and we see that pointwise convergence isn't sufficient to preserve Riemann-Stieltjes integration. We note that for $0 < \varepsilon < 1$ that f_n fails to converge uniformly to f on [0,1].

(3) Suppose that $f_n(x) = \dfrac{x}{n}$ $(n = 1, 2, 3, \ldots)$ on [0,1]. It follows that

$$\lim_{n\to\infty} f_n(x) = f(x) \equiv 0$$

and so $f_n \to f$ on [0,1]. Now, let $\varepsilon > 0$ be given. Find N so that $\dfrac{1}{N} < \varepsilon$. Then for $n \geq N$ and all $x \in [0,1]$ we have that $\left|f_n(x) - f(x)\right| = \dfrac{1}{n} \leq \dfrac{1}{N} < \varepsilon$. We conclude that $f_n \underset{unif}{\to} f$ on [0,1].

(4) For $n \geq 2$ define

$$f_n(x) = \begin{cases} n^2 x & \text{if } x \in [0, 1/n] \\ -n^2 (x - 2/n) & \text{if } x \in [1/n, 2/n] \\ 0 & \text{if } x \in [2/n, 1] \end{cases}.$$

Then for all $x \in [0,1]$ we have that $\lim_{n\to\infty} f_n(x) \equiv 0$. So, if $f(x) \equiv 0$ on [0,1], *then* $f_n \to f$ on [0,1]. Direct computation shows that

$$\int_0^1 f_n(x)\, dx = 1 \quad (n \geq 2)$$

and

$$\int_0^1 f(x)\ dx \equiv 0.$$

So, here even though $f_n \to f$ on [0,1] we have that

$$\int_a^b f\ d\alpha \neq \lim_{n\to\infty} \int_a^b f_n\ d\alpha.$$

That is, here we have that

$$\int_a^b \lim_{n\to\infty} f_n\ d\alpha \neq \lim_{n\to\infty} \int_a^b f_n\ d\alpha.$$

The next result is a simple consequence of Definition 90.

Theorem 91

Let $\left\{f_n\right\}_{n=1}^{\infty}$ be a sequence of functions defined on $[a,b]$ and let f be a function whose domain contains $[a,b]$. If $f_n \underset{unif}{\to} f$ on $[a,b]$, then $f_n \to f$ on $[a,b]$.

The above example shows that the converse of the above result is in general false. The next result gives sufficient conditions for the converse to be true.

Theorem 92 - Dini's Theorem

Let $\left\{f_n\right\}_{n=1}^{\infty}$ be a sequence of continuous functions defined on $[a,b]$ and let f be a function continuous whose domain contains $[a,b]$. If $\left\{f_n\right\}_{n=1}^{\infty}$ is a monotone sequence and if $f_n \to f$ on $[a,b]$, then $f_n \underset{unif}{\to} f$ there.

Observe that the continuity of f in Dini's Theorem is a part of the hypothesis.

The next result enables one under certain conditions to establish uniform convergence without actually knowing the limit function f in advance.

Theorem 93 - Cauchy's Criterion

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on $[a,b]$. Then $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$ iff for every $\varepsilon > 0$ there exists an integer N such that

$$\left| f_n(x) - f_m(x) \right| < \varepsilon$$

for all $x \in [a,b]$ and all $n, m \geq N$.

Theorem 94

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions defined on $[a,b]$ and let f be a function whose domain contains $[a,b]$. If $f_n \underset{unif}{\rightarrow} f$ on [a,b], then f is continuous there.

Part (4) of Example 20 shows that it is not always possible to interchange the two mathematical processes of integration and taking an ordinary limit. The next result provides sufficient conditions (which in fact are rather restrictive!) for

$$\int_a^b \lim_{n\to\infty} f_n \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha$$

to be true.

Theorem 38

Let α be a nondecreasing function on $[a,b]$ and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions which are integrable w/r/t α on $[a,b]$. If $f_n \underset{unif}{\rightarrow} f$ for some function f defined on [a,b], then

$$\int_a^b \lim_{n\to\infty} f_n \, d\alpha = \lim_{n\to\infty} \int_a^b f_n \, d\alpha$$

Example 21

Define $f_n(x) = x + \frac{x}{n}\sin nx$ for $n = 1, 2, 3, \ldots$ and let $\alpha(x) = x$.

(1) Use Theorem 38 to find $\int_0^{5\pi}\left(\lim_{n\to\infty} f_n\right) d\alpha$.

(2) Show directly that $\int_0^{5\pi}\lim_{n\to\infty} f_n \, d\alpha = \lim_{n\to\infty}\int_0^{5\pi} f_n \, d\alpha$.

Solution

(1) We first prove that $f_n(x) = x + \frac{x}{n}\sin nx \underset{unif}{\to} f(x) = x$. We see that

$$|f_n(x) - f(x)| = \left|\frac{x}{n}\sin nx\right| < \frac{5\pi}{n}$$

for all $x \in [0, 5\pi]$. Let $\varepsilon > 0$ be given. Find N so that $\frac{5\pi}{N} < \varepsilon$. Then for $n \geq N$ and all $x \in [0, 5\pi]$ we have that

$$|f_n(x) - f(x)| = \left|\frac{x}{n}\sin nx\right| < \frac{5\pi}{n} \leq \frac{5\pi}{N} < \varepsilon.$$

Thus, $f_n \underset{unif}{\to} f$ on $[0, 5\pi]$. So, by Theorem 38,

$$\int_0^{5\pi} f \, d\alpha = \int_0^{5\pi}\lim_{n\to\infty} f_n \, d\alpha = \lim_{n\to\infty}\int_0^{5\pi} f_n \, d\alpha$$

and we that

$$\int_0^{5\pi} x \, dx = \frac{25\pi^2}{2}.$$

(2) By the above, $\int_0^{5\pi}\lim_{n\to\infty} f_n \, d\alpha = \int_0^{5\pi} x \, dx = \frac{25\pi^2}{2}$. On the other hand,

$$\lim_{n\to\infty} \int_0^{5\pi} f_n \, d\alpha = \lim_{n\to\infty} \int_0^{5\pi} \left(x + \frac{x}{n} \sin xn \right) dx$$

$$= \lim_{n\to\infty} \left(\frac{x^2}{2} - \frac{x \cos nx}{n^2} + \frac{\sin nx}{n^3} \Big|_0^{5\pi} \right)$$

$$= \lim_{n\to\infty} \left(\frac{25\pi^2}{2} - \frac{5\pi \cos 5\pi n}{n^2} + \frac{\sin 5\pi n}{n^3} \right)$$

$$= \frac{25\pi^2}{2}$$

Thus, $\int_0^{5\pi} \lim_{n\to\infty} f_n \, d\alpha = \lim_{n\to\infty} \int_0^{5\pi} f_n \, d\alpha$.